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ABSTRACT

This paper is concerned with estimation and hypothesis testing of treatment effects in nonequivalent control group designs with the assumption that in the absence of treatment effects, natural growth conforms to a particular class of continuous growth models. Point estimation, interval estimation, and hypothesis testing procedures were developed for both treatment effects and differences in treatment effects. It was found that traditional methods of data analysis are incorrect under much of this class of growth models. (Author/PN)

The Estimation and Hypothesis Testing of Treatment Effects in Nonequivalent Control Group Designs When Continuous Growth Models are Assumed

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ABSTRACT

Both Lord (1967) and Cronbach and Furby (1970) have shown that there is no knowably correct method of analysis for data from a nonequivalent control group design unless some assumptions are made. The assumption made in this paper is that of a particular class of continuous growth models. It is shown that all of the traditional methods of data analysis are incorrect under much of this class of growth models. New methods of data analysis are then developed based on maximum likelihood estimation, jackknifing, and numerical analysis techniques.

This paper is concerned with estimation and hypothesis testing of treatment effects in nonequivalent control group designs (Campbell, 1969; Campbell & Stanley, 1966). A nonequivalent control group design is a quasi-experimental design in which the groups are formed by some method other than random assignment. Both Lord (1967) and Cronbach and Furby (1970) have argued that unless some assumptions are made there is no knowably correct method for the estimation and hypothesis testing of treatment effects for these designs. In this paper it is assumed that, in the absence of treatment effects, natural growth conforms to a particular class of continuous growth models.

This class of continuous growth models can be expressed symbolically as, for all times t,

$$Y^*_{ij}(t) = g_j(t) \cdot Y^*_{ij}(t_{1_j}) + h_j(t) + \alpha_j(t)$$

and (1)

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) \;; \quad j=1,2,\ldots,J$$

where J is the number of groups in the design;
$g_j(t)$ and $h_j(t)$ are continuous functions;
$\alpha_j(t)$ represents the population treatment effect for the *jth* group;
t_{1_j} is an arbitrary time point;

and $Y^*_{ij}(t)$, $Y_{ij}(t)$, and $e_{ij}(t)$ represent the true scores, observed scores, and errors of measurement, respectively, for the *ith* individual in the *jth* group, on the measure of interest.

Further assumptions are:

(1) Classical measurement theory holds. That is, for each time t, $Y_j^*(t)$ and $e_j(t)$ are uncorrelated and $E(e_{ij}(t)) = 0$.

and (2) Treatment effects are additive.

Notice that for this class of growth models, the correlation within each group between true scores at any two points in time is +1 . Further, notice that any number of groups are allowed under this class of growth models and that $g_j(t) \cdot Y_{ij}^*(t) + h_j(t)$ represents natural growth for these models. Finally, notice that the treatment effects, as defined by the $\alpha_j(t)$'s in the system of equations (1) are not the same as the usual definition of treatment effects. Let $\alpha(t)$ be the grand mean of the $\alpha_j(t)$'s. The usual definition of a treatment effect is given by $\alpha_j(t) - \alpha(t)$. Although it might be argued that this class of growth models is extremely restrictive, the authors (Blumberg & Porter, 1981) have shown that, in fact, a wide variety of natural growth patterns are included in this class (e.g. differential linear growth, exponential growth, and logistic growth).

The appropriateness, under the class of growth models described by the system of equations (1) of currently available methods of data analysis has been explored by Blumberg (in progress). She showed that ANOVA of Index of Response with K = g(t) (Cox, 1958), ANOVA of Standardized Change Scores with reliability correction (Kenny & Cohen, 1980) and ANOVA

of True Residual Gain Scores (Cronbach & Furby, 1970) yield unbiased estimates of differences in treatment effects and correctly test whether or not these differences are statistically significant when the system of equations (1) reduces to

$$Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(t_{1_j}) + h(t) + \alpha_j(t)$$

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t)$$

and

$$\sigma^*_{Y_j}(t) = \sigma^*_Y(t) \qquad \text{for all } j$$

where $\sigma^*_{Y_j}(t)$ represents the standard deviation of the true scores for the <u>jth</u> group.

Further, ANOVA of Raw Residual Gain Scores (Cronbach & Furby, 1970) and ANCOVA provide unbiased estimates of differences in treatment effects and correctly test for nonzero differences when

$$Y^*_{ij}(t) = g(t) \cdot Y^*_{ij}(t_{1_j}) + h(t) + \alpha_j(t)$$

$$\sigma^*_{Y_j}(t) = \sigma^*_Y(t) \qquad \text{for all } j$$

and

no errors of measurement are present in the data.

Other special situations were also found where one or more of the existing analysis strategies provide unbiased estimates of treatment effects and appropriate significance

testing procedures. Nevertheless, none of the conventional approaches to analyzing data from nonequivalent control group designs are appropriate when natural growth conforms to the system of equations (1) unless the $g_j(t)$'s are all equal and the $h_j(t)$'s are all equal. It should be pointed out here that the method of Empirical Bayes Estimation (Bryk, Strenio, & Weisberg, 1980; Strenio, Weisberg, & Bryk, in press) seems to be a very promising approach for the analysis of data arising from the application of nonequivalent control group designs under a wide variety of conditions. The method of Empirical Bayes Estimation, however, requires that the variance-covariance matrix of the true scores be non-singular. When the system of equations (1) is assumed, this variance-covariance matrix becomes singular and hence the method of Empirical Bayes Estimation cannot be used for the class of continuous growth models being considered here.

Hence, there is a need for new methods of data analysis which can provide point and interval estimates of and appropriate hypothesis testing procedures for treatment effects under the class of continuous growth models considered here. It is, however, not possible to develop methods which will work for the entire class of continuous growth models since the $h_j(t)$ and $\alpha_j(t)$ terms are confounded. Thus, when considering methods for estimating and testing treatment effects, assuming natural growth as defined in the system of equations (1), it is

necessary to distinguish between the following possibilities for the $h_j(t)$ terms:

(a) the exact natures of the $h_j(t)$'s are known (i.e., $h_1(t) = 3 \cdot t$, $h_2(t) = 4 \cdot t^2 + 3 \cdot t^{1/2}$, ... , $h_J(t) = \log_{10}(5 \cdot t^3 + 1)$);

(b) the functional forms of the $h_j(t)$'s are known (i.e., $h_1(t) = k_1 \cdot t$, $h_2(t) = k_2 \cdot t^2 + k_3 \cdot t^{1/2}$, ... , $h_J(t) = \log_{10}(k_4 \cdot t^3 + 1)$, where k_1, k_2, k_3, and k_4 are unknown real-valued constants); and

(c) for each time t, the $h_j(t)$'s are equal to some common value, say h(t) (i.e., for each t, $h_1(t) = h_2(t) = \ldots = h_J(t) = h(t)$).

The discussion of methods of data analysis for each of the above possibilities can be further broken down into six cases according to whether the exact natures of the $g_j(t)$'s are known, the functional forms of the $g_j(t)$'s are known, or nothing is known about the $g_j(t)$'s and according to whether or not errors of measurement are present (see Figure 1).

The remainder of this paper will present analysis procedures to be used for cases 2, 4, 6, 8, 10, and 12 (see Figure 1), since these are the cases which are likely to occur for data arising from educational research settings. Cases 14, 16, and 18 are also of interest, but the procedures to be used in these cases are sufficiently different that they merit a separate piece.

	$g_j(t)$'s known		Functional forms of $g_j(t)$'s known		Nothing known about $g_j(t)$'s	
	No Errors of Measurement	Errors of Measurement Present	No Errors of Measurement	Errors of Measurement Present	No Errors of Measurement	Errors of Measurement Present
$h_j(t)$'s known	Case 1	Case 2	Case 3	Case 4	Case 9	Case 10
Functional forms of $h_j(t)$'s known	Case 5	Case 6	Case 7	Case 8	Case 11	Case 12
$h_j(t) \equiv h(t)$	Case 13	Case 14	Case 15	Case 16	Case 17	Case 18

Figure 1

Subcases of the general growth model

Procedures have also been developed for the cases where no errors of measurement are present. The interested reader is referred to Blumberg (in progress) for the procedures for the cases not to be discussed here.

Overview of the Procedures

Each of the procedures to be discussed is a variation of a several stage method. At the first stage, estimates are obtained for the unknown constants (i.e., parameters) in the functional form expressions for the $g_j(t)$'s and/or $h_j(t)$'s. The method used to obtain the estimates of the parameters in the functional forms expressions for the $g_j(t)$'s and/or $h_j(t)$'s is the same whether one is estimating the parameters for the $g_j(t)$'s, for the $h_j(t)$'s, or for both the $g_j(t)$'s and $h_j(t)$'s. Hence, the discussion of the first stage will be done in general. The second stage concerns the estimation of treatment effects (i.e., the $\alpha_j(t)$'s in equation (1)), given the estimates obtained at the first stage and the particulars of the case of interest. Hence for this second stage, each of the cases must be discussed separately. The third stage concerns methods for interval estimation and hypothesis testing of both treatment effects and differences in treatment effects. These methods are the same for all the cases, once estimates of the $\alpha_j(t)$'s have been obtained. Therefore, the discussion of the third stage will be done for all 6 cases simultaneously.

Stage 1: Estimation of the $g_j(t)$'s and $h_j(t)$'s

The methods developed at stage 1 depend upon knowledge of each individual's scores on several pretests. Denote the times of the pretests for the jth group by $t_{1_j}, t_{2_j}, t_{3_j}, \ldots, t_{(p_j)_j}$, where p_j is the number of pretests. Hence, for the jth group, the system of equations (1) can be rewritten for the pretest times (remembering that $\alpha_j(t) \equiv 0$ for all pretest times) as

$$Y^*_{ij}(t_{k_j}) = g_j(t_{k_j}) \cdot Y^*_{ij}(t_{1_j}) + h_j(t_{k_j})$$

and (2)

$$Y_{ij}(t_{k_j}) = Y^*_{ij}(t_{k_j}) + e_{ij}(t_{k_j}) \quad ; \quad k=1,2,\ldots,p_j \; ; \quad i=1,2,\ldots,N_j$$

where N_j is the number of individuals in group j.

Stage 1 is divided into two substages. At the first substage, estimates of the $g_j(t_{k_j})$'s and $h_j(t_{k_j})$'s are obtained. At the second substage, the estimates of the $g_j(t_{k_j})$'s and $h_j(t_{k_j})$'s are used to obtain estimates of the unknown constants in the functional form expressions of the $g_j(t)$'s and/or $h_j(t)$'s.

For simplicity, the j subscript will be dropped for the remainder of the discussion of stage 1, since the estimation of the unknown constants in the functional form expressions for the $g_j(t)$'s and $h_j(t)$'s is done separately for each of the J groups. Dropping the j subscript from equation (2),

$$Y_i^*(t_k) = g(t_k) \cdot Y_i^*(t_1) + h(t_k) \tag{3}$$

and

$$Y_i(t_k) = Y_i^*(t_k) + e_i(t_k) \quad ; k=1,2,\ldots,p . \tag{4}$$

Equation (3) represents a linear structural (Madansky, 1959; Moran, 1971) or functional relation (DeGracie & Fuller, 1972; Lindley, 1947), where $g(t_k)$ and $h(t_k)$ are the slope and $Y_i^*(t_k)$-intercept, respectively, of the $Y_i^*(t_k)$ on $Y_i^*(t_1)$ regression line. There are several approaches that have been developed for finding estimates of the $g(t_k)$'s and $h(t_k)$'s. The approach that is to be used here is maximum likelihood estimation.

Estimation of the $g(t_k)$'s and $h(t_k)$'s

For a design with p pretests the system of equations represented by equations (3) and (4) can be described pictorially as in Figure 2.

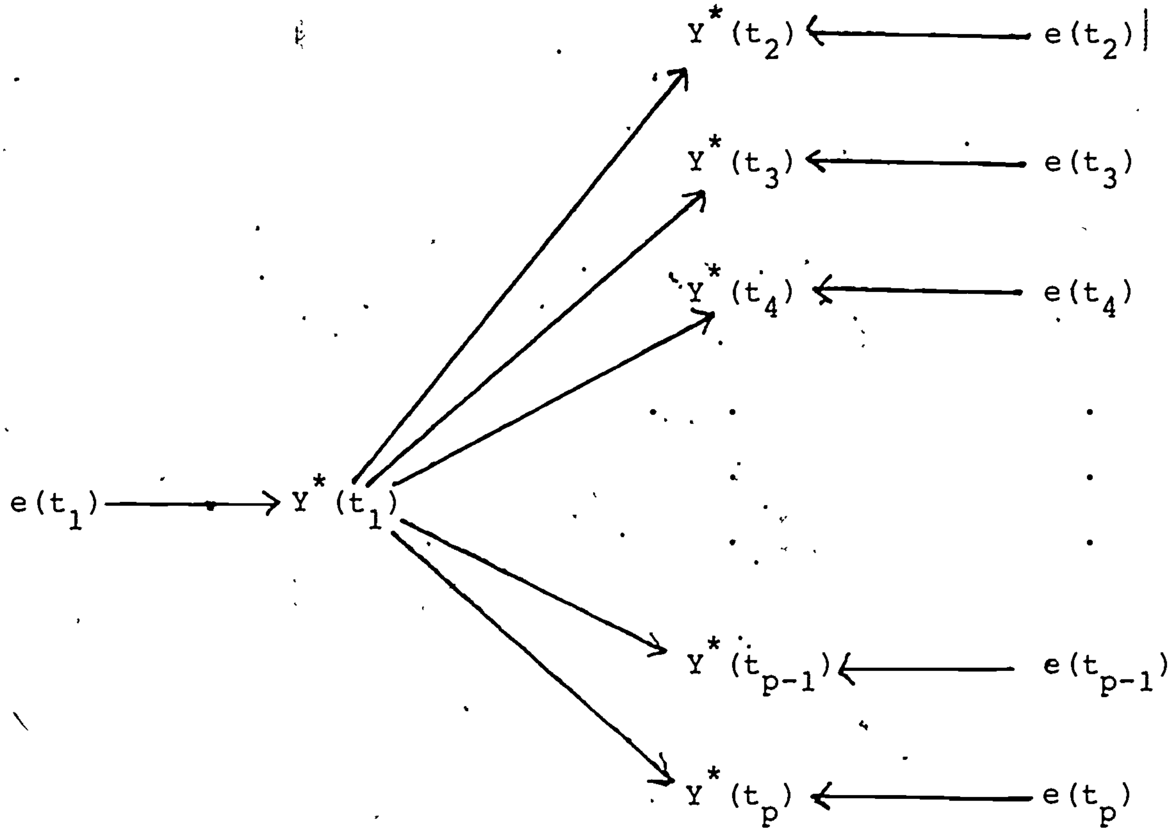

Figure 2

Pictorial representation of the structural relation

The system can be written in vector form as

$$\underline{Y_i^*(t_k)} = \underline{g(t_k)} \cdot Y_i^*(t_1) + \underline{h(t_k)} \qquad (5)$$

and

$$\underline{Y_i(t_k)} = \underline{Y_i^*(t_k)} + \underline{e_i(t_k)} \qquad (6)$$

Maximum likelihood requires expressions for the means, variances, and covariances of the observed variables under consideration. Taking the mean on both sides of equation (5) gives

$$\underline{\mu_Y(t_k)} = \underline{g(t_k)} \cdot \mu_Y(t_1) + \underline{h(t_k)} \tag{7}$$

where $\underline{\mu_Y(t_k)}$ is the vector of means. The variance of $Y(t_1)$ is given by

$$\sigma_Y^2(t_1) = [\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_1) \tag{8}$$

where $\sigma_e^2(t)$ represents the variance of the errors of measurement at time t. The variance of $Y(t_k)$ is given by

$$\sigma_Y^2(t_k) = [g(t_k)]^2 \cdot [\sigma_Y^*(t_1)]^2 + \sigma_e^2(t_k) \tag{9}$$

for k=1,2, ... , p . The covariance of $Y(t_k)$ and $Y(t_1)$ is given by

$$\sigma_{Y(t_k)Y(t_1)} = g(t_k) \cdot [\sigma_Y^*(t_1)]^2 \quad . \tag{10}$$

The covariance of $Y(t_k)$ and $Y(t_{k'})$, where k,k' = 2,3, ... , p and $k \neq k'$, is given by

$$\sigma_{Y(t_k)Y(t_{k'})} = g(t_k) \cdot g(t_{k'}) \cdot [\sigma_Y^*(t_1)]^2 \quad . \tag{11}$$

Consider the system of equations (7), (8), (9), (10), and (11) as one large system of equations. The maximum likelihood approach requires that this system be identifiable. A system of equations is said to be identifiable if and only if each of the unknown paramethers on the right hand side of the system can be expressed in terms of the unknown parameters on the left hand side of the system. For the models of continuous growth being considered here, the unknown parameters on the left hand side are $\mu_Y(t_1)$, $\mu_Y(t_2)$, ..., $\mu_Y(t_p)$ and the $\frac{p(p+1)}{2}$ parameters in the variance-covariance matrix of $Y(t_1)$, $Y(t_2)$, ... , $Y(t_p)$. The unknown parameters on the right hand side are $g(t_2)$, $g(t_3)$, ... , $g(t_p)$, $h(t_2)$, $h(t_3)$, ... , $h(t_p)$, $[\sigma^*_Y(t_1)]^2$, $\sigma^2_e(t_1)$, $\sigma^2_e(t_2)$, ... , $\sigma^2_e(t_{p-1})$, and $\sigma^2_e(t_p)$. The system of equations (7) to (11) is identifiable if and only if $p \geq 3$. When $p \geq 3$, the LISREL program (Jöreskog & Sörbom, 1978) can be used to obtain the maximum likelihood estimates of the parameters on the right hand side. When $p = 2$, additional assumptions must be made in order that the system of equations is identifiable.

Estimation of the Unknown Constants

Once the estimates of the $g(t_k)$'s and $h(t_k)$'s are obtained, estimates of the unknown constants in the general

functional form expressions for g(t) and h(t) can then be computed. For simplicity, the discussion here will be in terms of g(t). The method to be used for h(t) is analogous.

The method for determining estimates for the unknown constants in the expression for g(t) is dependent upon the functional form of g(t). For sake of illustration, consider the following two examples,

$$\text{a polynomial form:} \quad g(t) = 1 + \sum_{d=1}^{3} c_d \cdot (t - t_1)^d \qquad (12)$$

and

$$\text{an exponential form:} \quad g(t) = b \cdot c^{(t-t_1)} + (1 - b) \ . \qquad (13)$$

Polynomial Form

The object of the method to be described is to determine the values of c_1, c_2, and c_3 in the polynomial from knowledge of the $\widehat{g(t_k)}$'s, where the $\widehat{g(t_k)}$'s are the estimates of the $g(t_k)$'s obtained using the maximum likelihood approach. For simplicity, assume that $p = 5$, $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, $t_4 = 5$, and $t_5 = 7$. The method to be described will work for any $p \geq 4$ and for any set of values for the t_k's. Substituting $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, $t_4 = 5$, and $t_5 = 7$ into equation (12) yields

$$
\begin{aligned}
g(1) &= 1 + c_1 + c_2 + c_3 \\
g(4) &= 1 + 4c_1 + 16c_2 + 64c_3 \\
g(5) &= 1 + 5c_1 + 25c_2 + 125c_3 \qquad (14) \\
\text{and} \quad g(7) &= 1 + 7c_1 + 49c_2 + 343c_3 \ .
\end{aligned}
$$

But, the values of $g(1)$, $g(4)$, $g(5)$, and $g(7)$ are unknown. Therefore, the maximum likelihood estimates are used instead. Replacing $g(1)$, $g(4)$, $g(5)$, and $g(7)$ by $\widehat{g(1)}$, $\widehat{g(4)}$, $\widehat{g(5)}$, and $\widehat{g(7)}$ in the system of equations (14) yields

$$
\begin{aligned}
\widehat{g(1)} &= 1 + c_1 + c_2 + c_3 \\
\widehat{g(4)} &= 1 + 4c_1 + 16c_2 + 64c_3 \\
\widehat{g(5)} &= 1 + 5c_1 + 25c_2 + 125c_3 \qquad (15) \\
\text{and} \quad \widehat{g(7)} &= 1 + 7c_1 + 49c_2 + 343c_3 .
\end{aligned}
$$

The method of least squares is then used to obtain estimates of c_1, c_2 and c_3 from the system of equations (15).

Exponential Form

The object of the method to be described in this subsection is to determine the values of b and c in equation (13) from knowledge of the $\widehat{g(t_k)}$'s. The method to be described

will work for any $p \geq 3$ and for any set of values for the t_k's. Recall that the $\widehat{g(t_k)}$'s were already determined at the first substage. Next, the values of the t_k's and $\widehat{g(t_k)}$'s are substituted into equation (13) to yield a new system of equations analogous to system (15). For sake of illustration, let $p = 4$, $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, and $t_4 = 5$. Substituting these values into equation (13) yields

$$\widehat{g(1)} = b \cdot c + (1 - b)$$

$$\widehat{g(4)} = b \cdot c^4 + (1 - b) \qquad (16)$$

and

$$\widehat{g(5)} = b \cdot c^5 + (1 - b) \quad .$$

Estimates of b and c are then derived from the system of equations (16) using a combination of the numerical analysis techniques of least squares and the Newton-Raphson method.

Stage 2: Point Estimation of Treatment Effects

The discussion of the determination of the point estimators of treatment effects will be discussed separately for each case, since the process is slightly different in each case.

Case 2 estimates

For case 2 the exact natures of the $g_j(t)$'s and $h_j(t)$'s are assumed known. Unbiased point estimators of the treatment effects are given directly by

$$\widehat{\alpha_j(t)} = \overline{Y_j(t)} - [g_j(t) \cdot \overline{Y_j(t_{1_j})} + h_j(t)] \quad .$$

Notice that for this case, stage 1 is skipped completely.

Case 4 estimates

Case 4 assumes the exact natures of the $h_j(t)$'s are known but that only the functional forms of the $g_j(t)$'s are known. Recall that at stage 1, estimates were obtained for the unknown constants in the functional form expressions for the $g_j(t)$'s. For the jth group, let $\hat{g}_j(t)$ denote the function formed by substituting these estimates of the constants into the general functional form expression for $g_j(t)$. Point estimators of treatment effects are then given by

$$\widehat{\alpha_j(t)} = \overline{Y_j(t)} - [\hat{g}_j(t) \cdot \overline{Y_j(t_{1_j})} + h_j(t)] \quad .$$

Case 6 estimates

Case 6 assumes that the exact natures of the $g_j(t)$'s are known but that only the functional forms of the $h_j(t)$'s

are known. Analogous to case 4, for the jth group, form a new function, called $\hat{h}_j(t)$, by substituting the estimates of the unknown constants that were obtained in stage 1 into the functional form expression for $h_j(t)$. Point estimators of treatment effects are then given by

$$\widehat{\alpha_j(t)} = \overline{Y_j(t)} - [g_j(t) \cdot \overline{Y_j(t_{1_j})} + \hat{h}_j(t)] \quad .$$

Case 8 estimates

Case 8 assumes that only the functional forms of both the $g_j(t)$'s and $h_j(t)$'s are known. For this case, point estimators of treatment effects are given by

$$\widehat{\alpha_j(t)} = \overline{Y_j(t)} - [\hat{g}_j(t) \cdot \overline{Y_j(t_{1_j})} + \hat{h}_j(t)]$$

where the $\hat{g}_j(t)$'s and $\hat{h}_j(t)$'s are as defined previously.

Case 10 estimates

Case 10 assumes that the exact natures of the $h_j(t)$'s are known and that nothing is known about the $g_j(t)$'s. Define a new variable, $W_{ij}(t) = Y_{ij}(t) - h_j(t)$. Then, the system of equations (1) can be rewritten as

$$W^*_{ij}(t) = g_j(t) \cdot W^*_{ij}(t_{1_j}) + \alpha_j(t)$$

and (17)

$$W_{ij}(t) = W^*_{ij}(t) + e_{ij}(t) \quad .$$

Notice that the system of equations (17) is, for each j, a linear structural relation with a slope of $g_j(t)$ and a $W^*_j(t)$-intercept of $\alpha_j(t)$. Hence, for any particular time t, estimates of $g_j(t)$ and $\alpha_j(t)$ can be obtained directly by using maximum likelihood techniques, with $W_j(t_{1_j})$ as the independent variable and $W_j(t)$ as the dependent variable. It should be recalled here that in order to compute the maximum likelihood estimates in situations where there are only one independent and one dependent variable, it is necessary to make additional assumptions. The maximum likelihood estimates of the $\alpha_j(t)$'s are considered to be the stage 2 estimates and are labelled $\widehat{\alpha_j(t)}$'s.

Case 12 estimates

Case 12 assumes that only the functional forms of the $h_j(t)$'s are known and that nothing is known about the $g_j(t)$'s. In this case a new variable is formed by defining $U_{ij}(t) = Y_{ij}(t) - \hat{h}_j(t)$, where $\hat{h}_j(t)$ is defined as in case 6. The

method described for case 10 is then used to obtain the $\widehat{\alpha_j(t)}$'s by replacing the $W_{ij}(t)$'s of case 10 with the $U_{ij}(t)$'s.

In developing the point estimators of the treatment effects it was assumed that for any particular nonequivalent control group design, the known information about the growth curves for each of the J groups belonged to the same case. Since the determination of the point estimators is done separately for each group, insisting that all of the groups belong to the same case is overly restrictive. Hence, it will be assumed for the remainder of this paper that for each of the J groups, the known information about the growth curves allows the data analyst to place each of the groups into one of the cases 2,4,6,8,10, or 12. Once this is done, the $\widehat{\alpha_j(t)}$'s are derived separately for each group by using the methods given in this stage 2 section for the case to which the group's growth curves belong.

Bias of the Point Estimators

The procedures used to determine the point estimators of the treatment effects involved the use of maximum likelihood estimation at either stage 1 and/or stage 2, except for case 2 type growth curves. It is well known that maximum likelihood techniques often lead to biased estimators. These

estimators are, however, usually consistent (Patel, Kapadia, & Owen, 1976) and asymptotically efficient (Bickel & Doksum, 1977). Further, functions of consistent estimates are also usually consistent estimators. The asymptotic properties of the point estimators for treatment effects derived here for the various cases have not been studied in much detail. Further study of the asymptotic properties is needed.

Stage 3: Interval Estimation and Hypothesis Testing

In order to develop interval estimation and hypothesis testing procedures it is necessary to have estimates of the variances of the $\widehat{\alpha_j(t)}$'s. The method the authors have chosen to use to obtain estimates of the desired variances is jackknifing (Quenouille, 1956). The method of jackknifing was chosen for two reasons. First, it is the only practical method of obtaining variances for estimators that are formed by the use of a several-stage process such as those developed here. Second, and more important, the use of jackknifing usually leads to new point estimators which have a reduction in bias over the original point estimators, when the original estimators are biased.

The technique of jackknifing begins by drawing a random sample from a specified population. Let N denote the number of subjects in the sample. The N subjects are then divided

into m disjoint subsets, each of size $\frac{N}{m}$. Let γ be the parameter of interest and $\hat{\gamma}$ be an estimator of γ. Further, let $\hat{\gamma}^T$ be the value of $\hat{\gamma}$ when all N subjects are used, and let $\hat{\gamma}^{(\ell)}$ be the value of $\hat{\gamma}$ when the subsample of size $N - \frac{N}{m}$, where the ℓth subset has been deleted, is used.

Next define $J_\ell(\hat{\gamma})$ by

$$J_\ell(\hat{\gamma}) = m \cdot \hat{\gamma}^T - (m-1)\hat{\gamma}^{(\ell)} \quad ; \ \ell = 1, 2, \ldots, m$$

and define $J(\hat{\gamma})$ by

$$J(\hat{\gamma}) = \frac{1}{m} \cdot \sum_{\ell=1}^{m} J_\ell(\hat{\gamma}) \quad .$$

An estimate of the variance of $J(\hat{\gamma})$ is given by (Tukey, 1956)

$$S_J^2 = \frac{\sum_{\ell=1}^{m} [J_\ell(\hat{\gamma}) - J(\hat{\gamma})]^2}{m - 1} \quad .$$

Think of m as being fixed. Then as N approaches infinity, $\frac{N}{m}$ also approaches infinity. Gray and Schucany (1972) have shown that $\frac{\sqrt{m} \cdot [J(\hat{\gamma}) - \gamma]}{\sqrt{S_J^2}}$ is asymptotically distributed (as $\frac{N}{m} \rightarrow \infty$) as a random variable with $m - 1$ degrees of freedom.

The estimators of interest here are the $\widehat{\alpha_j(t)}$'s from stage 2. To apply the jackknifing technique, first divide the N_j subjects from the jth group into m_j distinct subsets. Next, define $\widehat{\alpha_j^{(\ell)}(t)}$ to be the value of the estimator, $\widehat{\alpha_j(t)}$, when the ℓth subset is deleted from the sample for the jth group. Once the $\widehat{\alpha_j^{(\ell)}(t)}$'s have been computed, new estimators of the $\alpha_j(t)$'s are given by

$$J(\widehat{\alpha_j(t)}) = \frac{1}{m_j} \sum_{\ell=1}^{m_j} J_\ell(\widehat{\alpha_j(t)}) \tag{18}$$

where

$$J_\ell(\widehat{\alpha_j(t)}) = m_j \cdot \widehat{\alpha_j(t)} - (m_j - 1) \cdot \widehat{\alpha_j^{(\ell)}(t)} .$$

Interval estimation and hypothesis testing procedures are available by observing that

$$\frac{\sqrt{m_j} \cdot [J(\widehat{\alpha_j(t)}) - \alpha_j(t)]}{\sqrt{(S_J^2(t))_j}}$$

is asymptotically distributed as a random variable with a Student's t distribution with $m_j - 1$ degrees of freedom

where $$(S_J^2(t))_j = \frac{\sum_{\ell=1}^{m_j} [J_\ell(\widehat{\alpha_j(t)}) - J(\widehat{\alpha_j(t)})]^2}{m_j - 1} .$$

An approximate $(1 - \alpha)\%$ confidence interval for $\alpha_j(t)$ (and hence, an α-level test for nonzero $\alpha_j(t)$) is then given by

$$J(\widehat{\alpha_j(t)}) \pm t_{m_j - 1}\left(1 - \frac{\alpha}{2}\right) \cdot \frac{\sqrt{(S_J^2(t))_j}}{\sqrt{m_j}} .$$

The estimation and hypothesis testing of differences in treatment effects are also of interest. A point estimator of $\alpha_j(t) - \alpha_{j'}(t)$, for two groups j and j', can be given by $J(\widehat{\alpha_j(t)}) - J(\widehat{\alpha_{j'}(t)})$ where $J(\widehat{\alpha_j(t)})$ and $J(\widehat{\alpha_{j'}(t)})$ are defined by equation (18). An approximate $(1-\alpha)\%$ test of the hypothesis of nonzero differences in treatment effects is accomplished by performing an ANOVA on the $J_\ell(\widehat{\alpha_j(t)})$'s. Notice that the unit of analysis for the ANOVA being performed here is the disjoint subsets formed in order to do the jackknifing and that the dependent variable is $J_\ell(\widehat{\alpha_j(t)})$.

Summary

In this paper point estimation, interval estimation, and hypothesis testing procedures were developed for both treatment effects and differences in treatment effects when

$$Y^*_{ij}(t) = g_j(t) \cdot Y^*_{ij}(t_{1_j}) + h_j(t) + \alpha_j(t)$$

and

$$Y_{ij}(t) = Y^*_{ij}(t) + e_{ij}(t) .$$

The only additional assumption needed was that for each of the J groups either the exact nature of or the functional form of $h_j(t)$ was known. Hence, techniques were developed which are applicable for a variety of natural growth situations where none of the presently available data analysis methods can be employed.

REFERENCES

Bickel, P.J., & Doksum, K.A. Mathematical statistics: Basic ideas and selected topics. San Francisco: Holden-Day, 1977.

Blumberg, C. J. The estimation and hypothesis testing of treatment effects in nonequivalent control group designs when continuous growth models are assumed. Unpublished Ph.D. dissertation, East Lansing: Michigan State University, in progress.

Blumberg, C. J., & Porter, A. C. Models of continuous growth and their implications for the analysis of nonequivalent control group designs. Paper presented at the American Educational Research Association annual meeting, Los Angeles, 1981.

Bryk, A. S., Strenio, J. F., & Weisberg, H. I. A method for estimating treatment effects when individuals are growing, Journal of Educational Statistics, 1980, 5, 5-34.

Campbell, D. T. Reforms as experiments. American Psychologist, 1969, 24, 409-429.

Campbell, D. T., & Stanley, J. C. Experimental and quasi-experimental designs for research. Chicago: Rand McNally, 1966.

Cox, D. R. Planning of experiments. New York: John Wiley & Sons, 1958.

Cronbach, L. J., & Furby, L. How we should measure 'change' or should we? Psychological Bulletin, 1970, 74, 68-80.

DeGracie, J. S., & Fuller, W. A. Estimation of the slope and analysis of covariance when the concomitant variable is measured with error. Journal of the American Statistical Association, 1972, 67, 930-937.

Gray, H. L., & Schucany, W. R. The generalized jackknife statistic. New York: Marcel Dekker, 1972.

Jöreskog, K. G., & Sörbom, D. LISREL IV: Analysis of linear structural relationships by the method of maximum likelihood. Chicago: International Educational Services, 1978.

Kenny, D. A., & Cohen, S. H. A reexamination of selection and growth processes in the nonequivalent control group design. In K. F. Schuessler (Ed.), Sociological Methodology 1980. Jossey-Bass, 1980.

Lindley, D. V. Regression lines and the linear functional relationship. Supplement Journal of Royal Statistical Society, 1947, 9, 218-244.

Lord, F. M. A paradox of interpretation of group comparisons. Psychological Bulletin, 1967, 68, 304-305.

Madansky, A. The fitting of straight lines when both variables are subject to error. Journal of the American Statistical Association, 1959, 54, 173-205.

Moran, P. A. P. Estimating structural and functional relationships. Journal of Multivariate Analysis, 1971, 1, 232-255.

Patel, J. K., Kapadia, C. H., & Owen, D. B. Handbook of Mathematical Distributions. New York: Marcel Dekker, 1976.

Quenouille, M. H. Notes on bias reduction. Biometrika, 1956, 43, 353-360.

Strenio, J. F., Weisberg, H. I., & Bryk, A. S. Empirical Bayes estimation of individual growth curve parameters and their relationship to covariates. Biometrika, in press.

Tukey, J. W. Bias and confidence in not-quite large samples (Abstract). Annals of Mathematical Statistics, 1958, 29, 614.